ODE/PDE Qual Questions Jan 2014 - Jan 2018

1 ODE

(Total distinct questions: 21. Format: do #1, choose 4 others.)

1.1 Jan 2014

1. (9/9)

   (a) State and prove the existence and uniqueness theorem for general systems $X' = f(t, X)$.

   (b) Find a function $f(X)$ not locally Lipschitz continuous and for which the system $X' = f(X)$, $X(0) = X_0$ does not have a unique solution. Provide details for the existence of multiple solutions.

   (c) Provide a detailed example of a system $X' = f(X)$ such that the interval of existence of solutions depend on the initial data. Can this example be a linear system?

(a) **Theorem.** *Let $X_0 \in U \subseteq \boldsymbol{R}^n$, where U is open, and let $I = (-b, b) \subseteq \boldsymbol{R}$. Suppose F is continuous on $I \times U$ and satisfies the Lipschitz estimate $|F(t, X_1) - F(t, X_2)| \leq L|X_1 - X_2|$ for $t \in I$ and $X_1, X_2 \in U$. Then there exists a unique solution to $\frac{dX}{dt} = F(t, X)$, $X(0) = X_0$, on some interval $J = (-a, a) \subseteq I$.*

Proof. First, observe that integrating $\frac{dX}{dt} = F(t, X)$ gives

$$X(t) = X_0 + \int_0^t F(s, X(s))ds.$$

Existence will follow from Picard iteration. Set $X_0(t) = X_0$ and recursively define

$$X_k(t) = X_0 + \int_0^t F(s, X_{k-1}(s))ds.$$

We will show that as $k \to \infty$, $X_k(t)$ converges to a unique solution. To do so, we will use the Contraction Mapping Thorem. Define $S = \left\{X(t) \in C(J, \mathbf{R}^n) \mid X(0) = X_0, \sup_{t \in J} \|X(t) - X_0\| \leq \varepsilon\right\}$, where $J = [-a, a]$, a to be chosen later, and ε chosen such that $\overline{B(X_0, \varepsilon)} \subseteq U$. S is a complete metric space under the metric $d(X(t), Y(t)) = \sup_{t \in J}|X(t) - Y(t)|$.

Define a map T on S by

$$(TX)(t) = X_0 + \int_0^t F(s, X(s))ds.$$

Now, $J \subseteq I$, so by the continuity of F and the Extreme Value Theorem, we set $M = \max_{J \times \overline{B(X_0, \varepsilon)}} |F(s, Y)|$. Choose $a < \frac{\varepsilon}{M}$. Then, $T : S \to S$, since $TX(0) = X_0$ and

$$|TX(t) - X_0| \leq \left|\int_0^t F(s, X(s))ds\right| \leq aM < a\frac{\varepsilon}{a} = \varepsilon.$$

By the Lipschitz hypothesis, if $t \in J$ and $Y, Z \in S$,

$$\begin{aligned}
|(TY)(t) - (TZ)(t)| &= \left| \int_0^t F(s, Y(s)) - F(s, Z(s)) ds \right| \\
&\leq \int_0^t |F(s, (Y(s)) - F(s, Z(s))| ds \\
&\leq \int_0^t L|Y(s) - Z(s)| ds \\
&\leq aL \sup_{s \in J} |Y(s) - Z(s)|.
\end{aligned}$$

If in addition, $a < \frac{1}{L}$, then $T : S \to S$ is a strict contraction, so by the Contraction Mapping Theorem, there exists a unique X such that

$$X(t) = X_0 + \int_0^t F(s, X(s)) ds.$$

□

(b) Consider $x' = 3x^{\frac{2}{3}}$, $x(0) = 0$. This example is not locally Lipschitz since at $x = 0$, $\left| \frac{3y^{\frac{2}{3}} - 0}{y - 0} \right| = \left| \frac{3}{y^{\frac{1}{3}}} \right| \not\leq L$ for y close to 0. And indeed, infinitely many solutions exist. For any $c > 0$,

$$x(t) = \begin{cases} 0 & \text{if } t \leq c; \\ (t-c)^3 & \text{if } t > c \end{cases}$$

solves it.

(c) Such an example cannot be linear, since we can solve every linear system. Indeed, let $f(X, t)$ be linear in X, so $f(X, t) = A(t)X + B(t)$. Then $f(X, t) - f(Y, t) = A(t)X + B(t) - A(t)Y - B(t) = A(t)(X - Y)$, so $|f(X, t) - f(Y, t)| \leq |A(t)||X - Y|$, and f is Lipschitz with constant $|A(t)|$. By the existence and uniqueness theorem, f has a unique solution.
For our example, consider $X' = 1 + X^2$, $X(0) = X_0$. Then $X(t) = \tan(t - c)$ for any $c = c(X_0)$. This is only continuous on $\frac{-\pi}{2} + c < t < \frac{\pi}{2} + c$.

2. (5/9)

Prove that if $F : \mathcal{O} \to \mathbf{R}^n$ is locally Lipschitz and $\mathcal{C} \subseteq \mathcal{O}$ is a compact set, then $F|_{\mathcal{C}}$ is Lipschitz.

Suppose not. Then for all $n \in \mathbf{N}$, there exists $x_n, y_n \in \mathcal{C}$ such that $|F(x_n) - F(y_n)| > n|x_n - y_n|$. Since $\mathcal{C}$ is compact, there exists a subsequence n_k such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$ for $x_0, y_0 \in \mathcal{C}$. Now,

$$|x_0 - y_0| = \lim_{n_k \to \infty} |x_{n_k} - y_{n_k}| \leq \lim_{n_k \to \infty} \frac{1}{n_k} |F(x_{n_k}) - F(y_{n_k})| \leq \lim_{n_k \to \infty} \frac{1}{n_k} 2 \sup_{\mathcal{C}} |F| = 0.$$

Thus, $x_0 = y_0$. Since F is locally Lipschitz, there exists U_{x_0} such that $x_0 \in U_{x_0} \subseteq \mathcal{O}$ with F Lipschitz on U_{x_0}. And by convergence, there exists N such that if $n_k \geq N$, then $x_{n_k}, y_{n_k} \in U_{x_0}$. Let L be the Lipschitz constant of F on U_{x_0}; then if $n_k > N, L$, we have $n_k|x_{n_k} - y_{n_k}| < |F(x_{n_k}) - F(y_{n_k})| \leq L|x_{n_k} - y_{n_k}|$, so $n_k \leq L$, a contradiction.

3. (5/9)

State and prove Grönwall's inequality.

Theorem. *Let $u : [0, a] \to \boldsymbol{R}$ be nonnegative and either*

- *$u'(t) \leq ku$, $u(0) = C$, $u \in C^1$, or*
- *$u(t) \leq C + \displaystyle\int_0^t ku(s)ds$, $u(0) = C$, $u \in C$.*

Then $u(t) \leq Ce^{kt}$.

Proof. If in the first case, we show $\frac{u(t)}{e^{kt}} \leq C$. See that $\frac{d}{dt}\left[ue^{-kt}\right] = e^{-kt}(u' - ku) \leq 0$, since $u' \leq ku$. Thus, ue^{-kt} is decreasing, so $ue^{-kt} \leq u(0) = C$.

If in the second case, let $v(t) = C + \displaystyle\int_0^t ku(s)ds$. Then $v'(t) = ku \leq kv$ and $v \in C^1$, so apply the first case. □

4. (3/9)

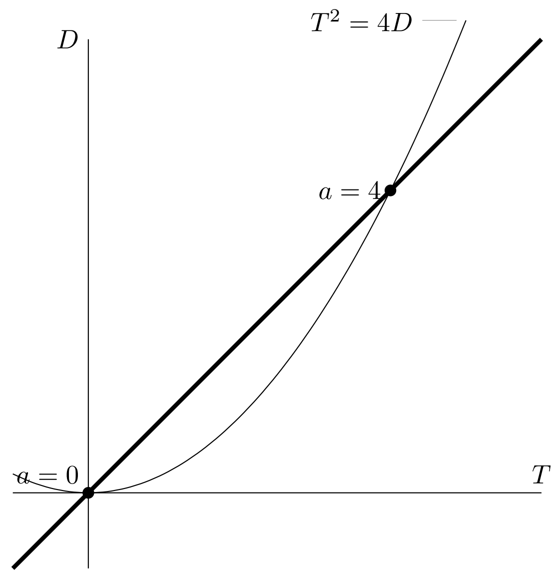
Consider a one-parameter family of linear systems given by

$$X' = \begin{bmatrix} a & a \\ -1 & 0 \end{bmatrix}.$$

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

(b) Discuss any bifurcations that occur along this path and compute the corresponding values of a.

(a) Notice that if $A = \begin{bmatrix} a & a \\ -1 & 0 \end{bmatrix}$, then $\operatorname{Tr} A = a$ and $\det A = a$. Thus in the trace-determinant plane, as a varies, we have



(b) See that the graph in (a) immediately shows us that if $a < 0$, the system is a saddle, if $0 < a < 4$, the system is a spiral source, and if $a > 4$, the system is a source. Let's also see this by checking eigenvalues. See that $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$, so $\lambda = \frac{a \pm \sqrt{a^2 - 4a}}{2} = \frac{a \pm \sqrt{a(a-4)}}{2}$. When $a < 0$, $\frac{a - \sqrt{a(a-4)}}{2}$ is negative, while $\frac{a + \sqrt{a(a-4)}}{2}$ is positive, so we have a saddle. When $0 < a < 4$, $\frac{a \pm \sqrt{a(a-4)}}{2}$ is

complex valued with real part $\frac{a}{2} > 0$, so we have a spiral source. Finally, when $a > 4$, $\frac{a+\sqrt{a(a-4)}}{2}$ and $\frac{a-\sqrt{a(a-4)}}{2}$ are both positive, so we have a source.
Bifurcations occur at $a = 0$ and $a = 4$.

5. (2/9)

Consider the family of differential equations

$$x' = ax - \sin x,$$

where a is a parameter ranging from $-\infty$ to ∞.

(a) Sketch the bifurcation diagram for this family of differential equations.

(b) Determine the qualitative behavior of all the bifurcations that occur as a increases from $-\infty$ to ∞.

6. (1/9)

Discuss the local and global behavior of solutions of

$$r' = r - r^3, \theta' = \sin^2\theta + a$$

at and near the bifurcation point $a = -1$. (For this problem, $x = r\cos\theta$ and $y = r\sin\theta$.

7. (5/9)

Find the general solution to

$$X' = AX + G(t)$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $G(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$.

8. (6/9)

A *Hamiltonian system* on $\mathbf{R}^2$ is a system of the form

$$x' = \frac{\partial H}{\partial y}(x, y)$$
$$y' = -\frac{\partial H}{\partial x}(x, y).$$

(a) Show that for a Hamiltonian system on $\mathbf{R}^2$, H is constant along every solution curve.

(b) Find a Hamiltonian function H for the system

$$x' = y$$
$$y' = -x^3 + x.$$

(a) Let $(x(t), y(t))$ be a solution curve for the system. By the chain rule,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0.$$

(b) Since $\frac{\partial H}{\partial y} = y$, $H(x,y) = \frac{y^2}{2} + f(x)$. Since $\frac{\partial H}{\partial x} = x^3 - x$, $H(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + g(y)$. Then, clearly, one such H is $H(x,y) = \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$. It is immediate that $\frac{\partial H}{\partial y} = y = x'$ and $\frac{\partial H}{\partial x} = x^3 - x = -\left(-x^3 + x\right) = -y'$.

9. (6/9)

Prove that the systems

$$X' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} X,$$

with $\lambda < 0$ and

$$X' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} X$$

are conjugate. (Do not invoke the theorem, explicitly prove the conjugacy for this special case).

1.2 Aug 2014

1. **1.1.1**
2. **1.1.2**
3. (5/9)

Let $\mathcal{O} \subseteq \mathbf{R}^n$ be open and suppose $F : \mathcal{O} \to \mathbf{R}^n$ has Lipschitz constant K. Let $Y(t)$ and $Z(t)$ be solutions of $X' = F(X)$ which remain in $\mathcal{O}$ and are defined on the interval $[t_0, t_1]$. Prove that for all $t \in [t_0, t_1]$,

$$|Y(t) - Z(t)| \le |Y(t_0) - Z(t_0)|e^{K(t-t_0)}.$$

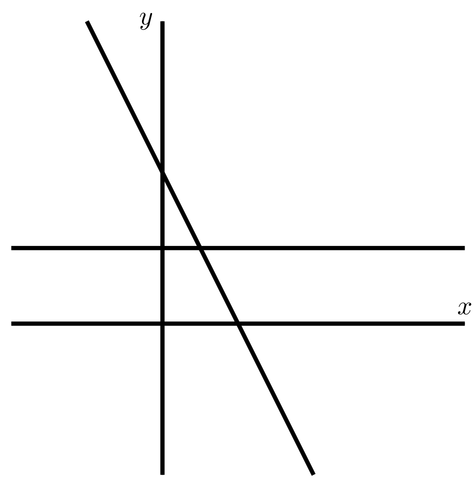
Write $\nu(t) = |Y(t) - Z(t)|$. Since $Y(t) - Z(t) = Y(t_0) - Z(t_0) + \int_{t_0}^{t} \Big(F(s, Y(s)) - F(s, Z(s))\Big)ds$ and F is Lipschitz with constant K, i.e., $|F(s, Y(s)) - F(s, Z(s))| \le K|Y(s) - Z(s)|$, this means $\nu(t) \le \nu(t_0) + \int_{t_0}^{t} K\nu(s)ds$. Now let $u(t) = \nu(t + t_0)$; thus $u(t) = \nu(t + t_0) \le \nu(t_0) + \int_{t_0}^{t+t_0} K\nu(s)ds = u(0) + \int_0^t Ku(s)ds$. So by Grönwall, $u(t) \le u(0)e^{Kt}$, so $\nu(t + t_0) \le \nu(t_0)e^{Kt}$, so $\nu(t) \le \nu(t_0)e^{K(t-t_0)}$, so $|Y(t) - Z(t)| \le |Y(t_0) - Z(t_0)|e^{K(t-t_0)}$, as desired.

4. (3/9)

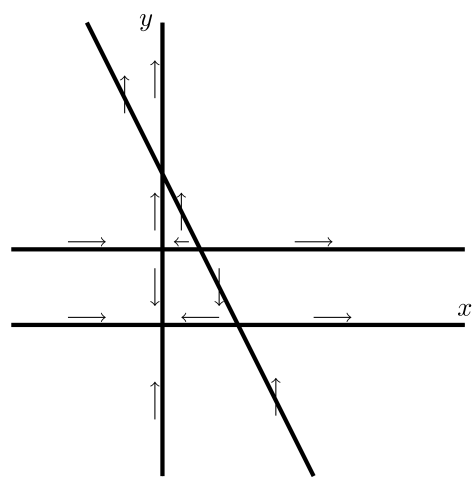
Sketch the x and y nullclines and use this information to determine the nature of the phase portrait for the system

$$x' = x(y + 2x - 2), y' = y(y - 1).$$

The nulclines are where $x' = 0$ and $y' = 0$. When $y' = y(y-1) = 0$, $y = 0$ or $y = 1$. When $x' = x(y + 2x - 2) = 0$, $x = 0$ or $y = -2x + 2$. The nulclines are therefore



We check signs to see the direction of the phase portrait across the nulclines. The resulting portrait is



The phase portrait can be completed by filling in arrows in the only way they can go.

5. (5/9)

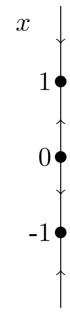
Consider the function $f(x) = x(1 - x^2)$.

(a) Sketch the phase line corresponding to the differential equation $x' = f(x)$.

(b) Let $g_a(x) = f(x) - ax$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$. Describe the bifurcations that occur in the family.

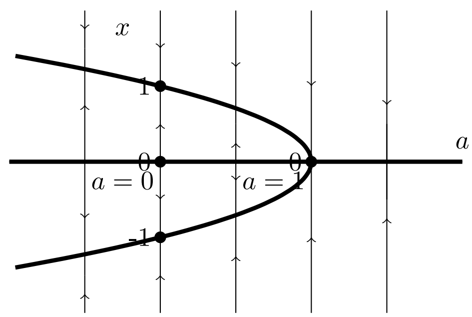
(c) Let $g_a(x) = f(x) + a$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$. Describe the bifurcations that occur in the family.

(a) Since $x' = x\left(1 - x^2\right) = x(1-x)(1+x)$, there are equilibria at $x = 0, 1, -1$. Choosing sample points and analyzing sign, the phase line is therefore



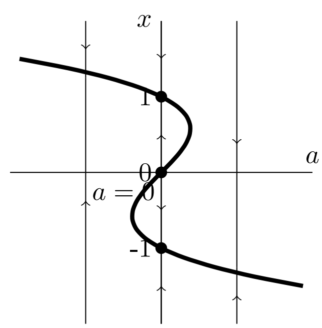
Notice that 1 and -1 are sinks, while 0 is a source.

(b) Here, $g_a(x) = x\left(1 - x^2\right) - ax = x\left(1 - x^2 - a\right)$. See that when $a = 0$, we have the phase line in part (a), and when $a = 1$, $g_1(x) = -x^3$, which has a single equilibrium at $x = 0$, a source. It is an easy exercise of picking a values in the regions $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ to see that the complete picture must be



A bifurcation occurs at $a = 1$, where we switch from three equilibria when $a < 1$ to a single sink when $a > 1$.

(c) Since $g_a(x) = x\left(1 - x^2\right) + a$, and varying a does not affect x, the bifurcation diagram will be



It only remains to find the bifurcation points, where the system switches from having one sink to three equilibria and from three equilibria to one sink. See that these points occur at the repeated roots of the cubic. In other words, to find a bifurcation point a, we solve $(x - \alpha)^2(x - \beta) = -x^3 + x + a$ for a. Let's equate coefficients; if $(x - \alpha)^2(x - \beta) = \left(x^2 - 2\alpha x + \alpha^2\right)(x - \beta) = x^3 - 2\alpha x^2 + \alpha^2 x - \beta x^2 + 2\alpha\beta x - \alpha^2\beta = x^3 - (2\alpha + \beta)x^2 + (\alpha^2 + 2\alpha\beta)x - \alpha^2\beta = -x^3 + x + a$, then

$$-x^3 + (2\alpha + \beta)x^2 - \left(\alpha^2 + 2\alpha\beta\right)x + \alpha^2\beta = -x^3 + x + a,$$

so

$$\begin{cases} 2\alpha + \beta = 0 \\ -\alpha^2 - 2\alpha\beta = 1 \\ \alpha^2\beta = a \end{cases}$$

So $\beta = -2\alpha$, and thus $-\alpha^2 - 2\alpha\beta = -\alpha^2 - 2\alpha(-2\alpha) = -\alpha^2 + 4\alpha^2 = 3\alpha^2 = 1$, so $\alpha = \pm\sqrt{\frac{1}{3}}$, $\beta = \mp 2\sqrt{\frac{1}{3}}$, and therefore $a = \frac{\mp 2}{3}\sqrt{\frac{1}{3}}$ are our bifurcation points.

6. (4/9)

For the nonlinear system $x' = \cos y$, $y' = \cos x$ on the set $\mathcal{O} = (0, 2\pi) \times (0, 2\pi)$,

(a) Find all of the equilibrium points in $\mathcal{O}$ and describe the behavior of the associated linearized systems.

(b) Sketch stable and unstable curves on $\mathcal{O}$ for any saddle points of this system.

7. (4/9)

Find the general solution and the phase portrait of the system

$$X' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

8. **1.1.8**

9. (6/9)

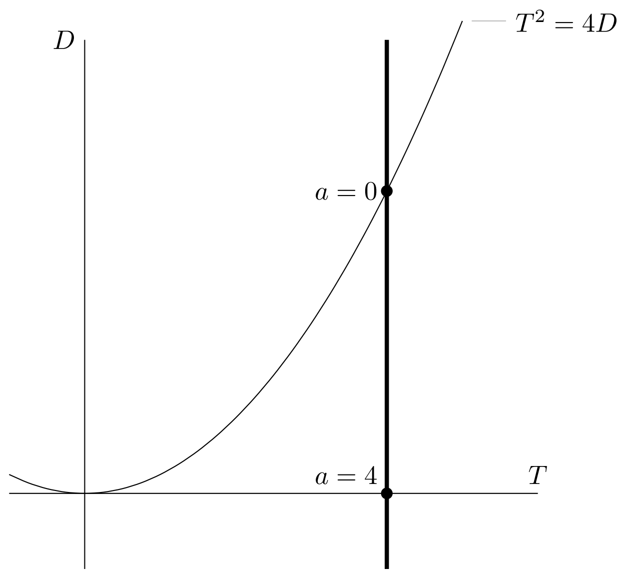
Consider a one parameter family of linear systems given by $X' = \begin{bmatrix} 2 & a \\ 1 & 2 \end{bmatrix} X$, where $a \in \mathbf{R}$.

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

(b) For which sets of a are the systems all topologically conjugate to each other? Classify the equilibrium points for each set (i.e., sink, source, saddle, or none of these).

(c) For which values of a is the system non-hyperbolic?

(a) See that if $A = \begin{bmatrix} 2 & a \\ 1 & 2 \end{bmatrix}$, then $\operatorname{Tr} A = 4$ and $\det A = 4 - a$, so in the trace-determinant plane, we see that as a varies, we get



(b) We can tell by the trace-determinant plane that when $a \in (4, \infty)$, the systems are all topologically conjugate to each other and saddles, when $a \in (0, 4)$, the systems are all topologically conjugate and sources, and when $a \in (-\infty, 0)$, the systems are all topologically conjugate and spiral sources, but let us show this using eigenvalues as well. Since $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$, we have $\lambda = \frac{4 \pm \sqrt{4^2 - 4(4-a)}}{2} = \frac{4 \pm \sqrt{4a}}{2} = 2 \pm \sqrt{a}$. When $a > 4$, we have two real eigenvalues, one positive and one negative, so the system is a saddle. When $0 < a < 4$, both $2 + \sqrt{a}$ and $2 - \sqrt{a}$ are positive, so we have sources. Finally, when $a < 0$, we have complex eigenvalues with real part $2 > 0$, so we get spiral sources, all as claimed.

(c) The system is non-hyperbolic when $a = 0$ and $a = 4$.

1.3 Jan 2015

1. **1.1.1**
2. **1.2.3**
3. **1.1.3**
4. **1.2.9**
5. **1.1.5**
6. **1.2.6**
7. **1.1.7**
8. **1.1.8**
9. **1.1.9**

1.4 Aug 2015

1. **1.1.1**
2. **1.1.2**
3. **1.1.3**
4. **1.1.4**
5. (5/9)

Consider the first-order nonautonomous equation with $x' = a(t)x$, where $a(t)$ is differentiable and periodic with period T. Compute the Poincaré map for this equation. Prove that all solutions of this equation are periodic with period T if and only if

$$\int_0^T a(s)ds = 0.$$

First we find the Poincaré map. Observe as

$$\begin{aligned}
x' &= a(t)x \\
\frac{x'}{x} &= a(t) \\
\int_0^t \frac{x'}{x} ds &= \int_0^t a(s)ds \\
\ln(x(t)) - \ln(x(0)) &= \int_0^t a(s)ds \\
\ln\left(\frac{x(t)}{x(0)}\right) &= \int_0^t a(s)ds \\
\frac{x(t)}{x(0)} &= \exp\left(\int_0^t a(s)ds\right) \\
x(t) &= x(0)\exp\left(\int_0^t a(s)ds\right).
\end{aligned}$$

Then $\varphi(x(0),0) = x(0)\exp\left(\int_0^0 a(s)ds\right) = x(0)$ and $\varphi(x(0),T) = x(0)\exp\left(\int_0^T a(s)ds\right) = p(x(0))$, so $\varphi(x,t) = x\exp\left(\int_0^t a(s)ds\right)$.

To show that all solutions are periodic with period T if and only if $\int_0^T a(s)ds = 0$, we let x be an arbitrary periodic solution with period T. Then $\varphi(x,T) - \varphi(x,0) = 0$ if and only if $x\exp\left(\int_0^T a(s)ds\right) - x = 0$, so $x\left(\exp\left(\int_0^T a(s)ds\right) - 1\right) = 0$, so $\exp\left(\int_0^T a(s)ds\right) = 1$, so $\int_0^T a(s)ds = 0$, as desired.

6. **1.2.4**
7. **1.2.7**
8. **1.1.8**
9. **1.1.9**

1.5 Jan 2016

1. **1.1.1**
2. **1.2.3**
3. (2/9)

Prove the following general fact: if $C \geq 0$ and $u, v : [0,\beta] \to \mathbf{R}$ are continuous and nonnegative, and

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0,\beta]$, then $u(t) \leq Ce^{V(t)}$ where

$$V(t) = \int_0^t v(s)ds.$$

Let $h(t) = C + \int_0^t u(s)v(s)ds$, so by assumption, $u(t) \leq h(t)$. By construction, $h'(t) = u(t)v(t)$. Then $\frac{h'(t)}{h(t)} = \frac{u(t)v(t)}{h(t)} \leq \frac{h(t)v(t)}{h(t)} = v(t)$, so $\frac{d}{dt}\left[\ln(h(t))\right] \leq v(t)$. As u and v are nonnegative,

$$\begin{aligned}
\int_0^t \frac{d}{dt}\left[\ln(h(s))\right] ds &\leq \int_0^t v(s)ds = V(t) \\
\ln(h(t)) - \ln(h(0)) &\leq V(t) \\
\ln\left(\frac{h(t)}{h(0)}\right) &\leq V(t) \\
\frac{h(t)}{h(0)} &\leq e^{V(t)} \\
h(t) &\leq h(0)e^{V(t)} \\
u(t) \leq h(t) &\leq Ce^{V(t)},
\end{aligned}$$

as desired.

4. **1.2.9**
5. **1.4.5**
6. **1.2.6**
7. **1.1.7**
8. **1.2.5**
9. **1.1.9**

1.6 Aug 2016

1. **1.1.1**
2. **1.1.2**
3. **1.1.3**
4. **1.2.9**
5. **1.4.5**
6. (2/9)

   For each of the following non-linear systems find the equilibrium points and describe the behavior of the associated linearized system. Describe the phase portrait of the non-linear system and compare its solutions with the solution of the linearized system near the equilibrium point.

   $$x' = x + y^2, y' = y$$
   $$x' = x\left(x^2 + y^2\right), y' = y\left(x^2 + y^2\right).$$

7. **1.2.7**
8. **1.2.5**
9. **1.1.9**

1.7 Jan 2017

1. **1.1.1**
2. **1.1.2**
3. **1.5.3**
4. **1.2.9**
5. **1.4.5**
6. **1.1.8**
7. **1.1.7**
8. **1.2.6**
9. (Warning: **1.2.5**, with a +!)

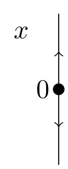
Consider the function $f(x) = x\left(1 + x^2\right)$.

(a) Sketch the phase line corresponding to the differential equation $x' = f(x)$.

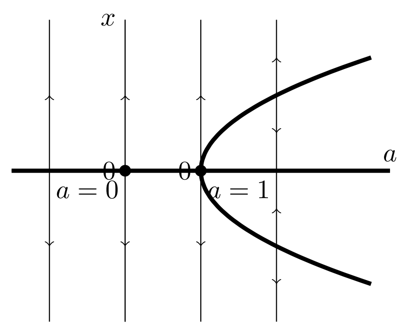
(b) Let $g_a(x) = f(x) - ax$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$. Describe the bifurcations that occur in the family.

(c) Let $g_a(x) = f(x) + a$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$. Describe the bifurcations that occur in the family.

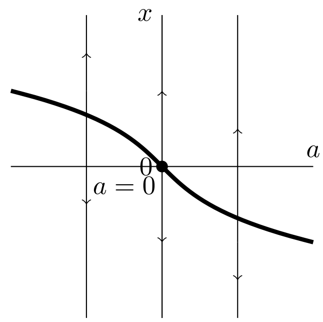
(a) Note that actually the + sign makes a significant difference in several parts, specifically because $1 + x^2$ no longer factors in the reals. For part (a), $x = 0$ is the only equilibrium, and the phase line is



(b) When $g_a(x) = x\left(1 + x^2\right) - ax = x\left(1 + x^2 - a\right)$, see that we have almost the same behavior as **1.2.5**. Choosing $a = 1$, $g_1(x) = x^3$, and for larger a, the quadratic piece now factors. So see that the bifurcation diagram must be



(c) Here, $g_a(x) = x\left(1 + x^2\right) + a = x^3 + x + a$. Since $x^3 + x : \mathbf{R} \to \mathbf{R}$ is bijective, this means there are no repeated roots, and thus no bifurcation points; the bifurcation diagram is simply



1.8 Aug 2017

1. **1.1.1**
2. **1.1.3**
3. **1.2.9**
4. **1.2.3**
5. **1.2.4**
6. **1.6.6**
7. **1.2.7**
8. **1.2.5**
9. **1.1.8**

1.9 Jan 2018

1. **1.1.1**
2. **1.1.3**
3. **1.1.4**
4. **1.2.3**
5. **1.1.9**
6. (1/9)

   For the nonlinear system $x' = \sin x$, $y' = \cos y$ on the set $\mathcal{O} = \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \times (0, 2\pi)$, describe the phase portrait (carefully justifying the behavior of the system near equilibrium points). Clearly identify any stable or unstable curves.

7. **1.1.7**
8. **1.4.5**
9. (1/9)

   Sketch the phase portrait of the Hamiltonian system $x' = x^2 - 2xy$, $y' = y^2 - 2xy$. Be sure to justify your sketch near any non-hyperbolic points.

If $\frac{\partial H}{\partial y} = x' = x^2 - 2xy$, then $H(x,y) = x^2y - xy^2 + f(x)$, and if $\frac{\partial H}{\partial x} = -y' = -y^2 + 2xy$, then $H(x,y) = -y^2x + x^2y + g(y)$, so $H(x,y) = x^2y - xy^2$. If $H(x,y) = 0$, then $xy^2 = x^2y$, so we have level curves $x = 0$, $y = 0$, and $y = x$.

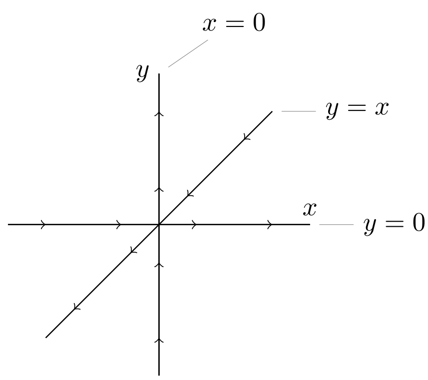
To analyze equilibria, see that $x' = y' = 0$ if and only if $(x,y) = (0,0)$. If we consider the linearized system at $(0,0)$, we have

$$DF = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x-2y & -2x \\ -2y & 2y-2x \end{bmatrix},$$

so

$$DF|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the linearized system at $(0,0)$ is non-hyperbolic, and we gain no information. We thus analyze direction on level curves. See that if $x = 0$, then $x' = 0$ and $y' > 0$, if $y = 0$, then $x' > 0$ and $y' = 0$, and if $y = x$, then $x' < 0$ and $y' < 0$. We have the following phase diagram.



Solution curves can be filled in as desired.

2 PDE

(Total distinct questions: 24. Format: do 5, at least one from each group.)

2.1 Jan 2014

I. Laplace Equation

1. (5/9)

   > Let (u_n) be a sequence of harmonic functions in $\Omega \subseteq \mathbf{R}^n$ where Ω is a bounded, open set. If u_n converges uniformly on compact subsets of Ω, then is the limit function u a harmonic function? Provide an argument or a counterexample.

   Yes, the limit function u must be harmonic. We provide a proof. As each u_n is harmonic, it satisfies the mean value property; i.e., $u_n(x) = ⨍_{\partial B(x,r)} u_n d\sigma$ for every $\overline{B(x,r)} \subseteq \Omega$. By uniform convergence in $\sigma(y)$ on $\overline{B(x,r)}$, we can interchange limits; thus, we must have that

   $$u = \lim u_n = \lim ⨍_{\partial B(x,r)} u_n d\sigma = ⨍_{\partial B(x,r)} \lim u_n d\sigma = ⨍_{\partial B(x,r)} u d\sigma.$$

   Since this holds for every $\overline{B(x,r)} \subseteq \Omega$, by the converse to the mean value property, u is harmonic.

2. (5/9)

This problem has two parts.

(a) State and prove the Harnack inequality for harmonic functions in an open set of $\mathbf{R}^n$.

(b) Use the Harnack inequality to answer the following questions: Is it possible to find a nonnegative harmonic function f defined in a neighborhood of $0 \in \mathbf{R}^n$ satisfying $f(0) = 0$ and so that f is not identically 0? What is the answer if we remove the sign condition?

(a) **Theorem.** *If u is a nonnegative harmonic function on $U \subset\subset \Omega$, connected and open, then there exists a positive constant C such that $\sup_U u \leq C \inf_U u$.*

Proof. Let $r = \frac{1}{4}\operatorname{dist}(U, \partial\Omega)$. Choose $x, y \in U$ so that $|x - y| < r$. Then

$$u(x) = \fint_{B(x,2r)} udz = \frac{1}{\omega_n 2^n r^n}\int_{B(x,2r)} udz \geq \frac{1}{\omega_n 2^n r^n}\int_{B(y,r)} udz = \frac{1}{2^n}\fint_{B(y,r)} udz = \frac{1}{2^n}u(y).$$

Thus, $2^n u(x) \geq u(y)$ if $|x - y| < r$. Since U is connected and $\overline{U}$ is compact, cover $\overline{U}$ by a finite chain of balls $\{B_j \mid j = 1, ..., N\}$, each with radius $\frac{r}{2}$ and $B_j \cap B_{j-1} \neq \emptyset$ for $j = 2, ..., N$. Thus, $u(y) \leq 2^{nN} u(x)$ for all $x, y \in U$. $\square$

(b) It is not possible. Let U be a neighborhood of 0 such that $f \geq 0$ on U. Since f is harmonic and nonnegative, by Harnack, there exists C such that $\sup_U f \leq C \inf_U f$. Since f is nonnegative and $f(0) = 0$, $\inf_U f = 0$. Thus $\sup_U f = 0$, and $f \equiv 0$.

On the other hand, if we remove the sign condition, an easy example arises by letting $n = 1$ and considering $f(x) = x$. Then, $\Delta f \equiv 0$, so f is harmonic, and around any neighborhood of 0, $f(0) = 0$ but $f \not\equiv 0$.

3. (5/9)

This problem is divided into two parts:

(a) State and prove the mean value property for harmonic functions.

(b) State and prove the strong maximum principle for harmonic functions.

(a) **Theorem.** *Let $U \subseteq \boldsymbol{R}^n$ be open. If $u \in C^2(U)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} ud\sigma = \fint_{B(x,r)} udy$$

for every $\overline{B(x,r)} \subseteq U$.

Proof. Set $\varphi(r) = \fint_{\partial B(x,r)} u(y)d\sigma(y)$. So

$$\varphi(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} u(y)d\sigma(y) = \frac{1}{n\omega_n}\int_{\partial B(x,r)} u(x+rz)d\sigma(z) = \fint_{\partial B(0,1)} u(x+rz)d\sigma(z).$$

Now, see that

$$\begin{aligned}
\varphi'(r) &= \fint_{\partial B(0,1)} \frac{\partial}{\partial r} u(x+rz)d\sigma(z) \\
&= \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot z d\sigma(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r} d\sigma(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\cdot \nu(y) d\sigma(y) \\
&= \fint_{\partial B(x,r)} \frac{du(y)}{d\nu} d\sigma(y) \\
&= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{du}{d\nu} d\sigma(y) \\
&= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \Delta u(y) dy \\
&= 0,
\end{aligned}$$

so $\varphi(r)$ is constant. Thus

$$\varphi(0) = \lim_{t\to 0^+} \varphi(t) = \lim_{t\to 0^+} \fint_{\partial B(x,t)} u(y)d\sigma(y) = u(x),$$

so $u(x) = \fint_{\partial B(x,r)} u(y)d\sigma(y)$. Finally, see that

$$\int_{B(x,r)} u dy = \int_0^r \left(\int_{\partial B(x,r)} u d\sigma\right) ds = \int_0^r n\omega_n s^{-n-1} u(x) ds = u(x) n\omega_n \frac{s^n}{n}\bigg|_0^n = u(x)|B(x,r)|,$$

so divide by $|B(x,r)|$ to get the result. □

(b) **Theorem.** *Suppose $u \in C\left(\overline{U}\right)$ is harmonic on U. Then $\max_U u = \max_{\partial U} u$, and if U is connected and there exists a point $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u$, then u is constant on U.*

Proof. Suppose there exists $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u = M$. Then by the mean value property, $M = u(x_0) = \fint_{B(x_0,r)} u dy \leq M$. Equality can hold only if $u \equiv M$, so $u(y) = M$ for all $y \in B(x_0, r)$. Thus the set $\{x \in U \mid u(x) = M\}$ is clopen in U, so if U is connected, all of U. If U is not connected, U is a union of connected components, so the first part follows by applying the argument to each connected component. □

II. Heat Equations

4. (4/9)

Denote by $B(0,1)$ the unit ball in $\mathbf{R}^n$. Let $g : \overline{B(0,1)} \to \mathbf{R}$ be a continuous function such that $g(x) = 1$ for $x \in \partial B(0,1)$. Let $u \in C^\infty(B(0,1) \times [0,\infty))$ be caloric in $B(0,1) \times (0,\infty)$ and satisfy boundary/initial data:

$$\begin{cases} u(x,0) = g(x) & x \in B(0,1), \\ u(x,t) = 1 & x \in \partial B(0,1), t > 0 \end{cases}$$

Prove that the energy $E(t) = \int_{B(0,1)} |\nabla u(x,t)|^2 \, dx$ is a non-increasing function of t.

See that

$$\begin{aligned} \frac{d}{dt}[E(t)] &= \frac{d}{dt}\left[\int_{B(0,1)} |\nabla u(x,t)|^2 dx\right] \\ &= \int_{B(0,1)} 2\nabla_x u \cdot \nabla_t u_t dx \\ &= -2\int_{B(0,1)} \Delta_x u \cdot u_t dt + 2\int_{\partial B(0,1)} \frac{du}{d\nu} \cdot \frac{du}{dt} \sigma(x). \end{aligned}$$

But on $\partial B(0,1)$, $u(x,t) \equiv 1$. This means $\frac{du}{dt} \equiv 0$ on $\partial B(0,1)$, so

$$\frac{d}{dt}[E(t)] = -2\int_{B(0,1)} \Delta_x \cdot u_t dt.$$

Since u is caloric, $u_t = \Delta_x u$, so

$$\frac{d}{dt}[E(t)] = -2\int_{B(0,1)} \Delta_x u \cdot \Delta_x u dx = -2\int_{B(0,1)} |\Delta_x u|^2 dx \leq 0,$$

so $E(t)$ is non-increasing, as desired.

5. (6/9)

This problem is divided in two parts:

(a) Let $\varphi : \mathbf{R} \to \mathbf{R}$ be smooth and convex. Show that if u is caloric, then $\varphi(u)$ is sub-caloric.

(b) If u is a caloric function, show that $v = |\nabla u|^2 + {u_t}^2$ is sub-caloric (i.e., $\partial_t v - \Delta v \leq 0$).

(a) First note that

$$\begin{aligned} \frac{\partial}{\partial x_i}\Big[\varphi(u)\Big] &= \varphi'(u) \cdot \frac{\partial u}{\partial x_i}, \text{ so} \\ \frac{\partial^2}{\partial {x_i}^2}\Big[\varphi(u)\Big] &= \varphi''(u) \cdot \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} + \varphi'(u) \cdot \frac{\partial^2 u}{\partial {x_i}^2}, \text{ so} \\ \Delta\Big[\varphi(u)\Big] &= \sum_{i=1}^n \left(\varphi''(u) \cdot \left(\frac{\partial u}{\partial x_i}\right)^2 + \varphi'(u) \cdot \frac{\partial^2 u}{\partial {x_i}^2}\right) \\ &= \sum_{i=1}^n \varphi''(u) \cdot \left(\frac{\partial u}{\partial x_i}\right)^2 + \varphi'(u) \cdot \Delta u. \end{aligned}$$

Then, as u is caloric, $\frac{\partial u}{\partial t} - \Delta u = 0$, and

$$\begin{aligned}\big(\varphi(u)\big)_t - \Delta\big(\varphi(u)\big) &= \varphi'(u)\cdot\frac{\partial u}{\partial t} - \sum_{i=1}^{n}\varphi''(u)\cdot\left(\frac{\partial u}{\partial x_i}\right)^2 - \varphi'(u)\cdot\Delta u \\ &= \varphi'(u)\cdot\left(\frac{\partial u}{\partial t} - \Delta u\right) - \sum_{i=1}^{n}\varphi''(u)\cdot\left(\frac{\partial u}{\partial x_i}\right)^2 \\ &= \varphi'(u)\cdot(0) - \sum_{i=1}^{n}\varphi''(u)\cdot\left(\frac{\partial u}{\partial x_i}\right)^2 \\ &= -\sum_{i=1}^{n}\varphi''(u)\cdot\left(\frac{\partial u}{\partial x_i}\right)^2 \\ &\leq 0,\end{aligned}$$

since φ convex means $\varphi''(u) \geq 0$.

(b) Notice that v can be thought of as a function of u; i.e., $v(u) = |\nabla u|^2 + {u_t}^2$. Notice that $\nabla[\cdot]$ and $\frac{\partial}{\partial t}[\cdot]$ are both linear operations, hence convex, that $|\cdot|^2$ and $\cdot^2$ are both convex, and that the summation of convex functions is convex. Therefore, $|\nabla[\cdot]|^2 + \left(\frac{\partial}{\partial t}[\cdot]\right)^2$ is a smooth, convex function, so by part (a), since u is caloric, $v = v(u)$ is sub-caloric.

III. Wave Equations

6. (1/9)

> Suppose that u is a C^2 solution of the wave equation in $\mathbf{R}^n \times \mathbf{R}$. Show that if $u(\cdot, t_0)$ has compact support in $\mathbf{R}^n$ for some t_0, then $u(\cdot, t)$ has compact support in $\mathbf{R}^n$ for all t.

This is **false**. See Homework 5.

7. (2/9)

> (Stokes' Rule) Assume u solves the initial value problem
>
> $$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ u = 0, u_t = h & \text{on } \mathbf{R}^n \times \{t = 0\} \end{cases}$$
>
> where $h \in C_C^\infty(\mathbf{R}^n)$. Show that $v = u_t$ solves
>
> $$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ v = h, v_t = 0 & \text{on } \mathbf{R}^n \times \{t = 0\}. \end{cases}$$

If $u_{tt} - \Delta u = 0$, then, differentiating in t, we have $u_{ttt} - (\Delta u)_t = 0$. Since $u \in C^2$, $(\Delta u)_t = \Delta(u_t)$, and since $v = u_t$, we have that $v_{tt} - \Delta v = 0$ in $\mathbf{R}^n \times (0,\infty)$. Furthermore, since $u_t = h$ on $\mathbf{R}^n \times \{t = 0\}$, $v = h$ on $\mathbf{R}^n \times \{t = 0\}$. Finally, we must show that $v_t(x, 0) = 0$. See that

$$v_t(x,0) = u_{tt}(x,0) = \lim_{t\to 0^+} u_{tt}(x,t) = \lim_{t\to 0^+} \Delta u(x,t) = \Delta u(x,0) = \Delta(0) = 0,$$

as desired.

IV. Miscellaneous

8. (4/9)

Euler's PDE for a homogeneous function $u(x_1, ..., x_n)$ is

$$\sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k} = \alpha u$$

where $\alpha \neq 0$ is a constant.

(a) Solve Euler's PDE with initial condition $u(x_1, ..., x_{n-1}, 1) = h(x_1, ..., x_{n-1})$ for some function $h : \mathbf{R}^{n-1} \to \mathbf{R}$.

(b) A function g is homogeneous of order β if $g(\lambda x) = \lambda^\beta g(x)$. Determine to what degree your solution u from (a) is homogeneous.

9. (9/9)

If $(1-\Delta)^3 u = f$ and $u \in L^2(\mathbf{R}^n)$, $f \in H^s(\mathbf{R}^n)$ for some $s \geq 0$, use the Fourier transform to show directly that $u \in H^{s+6}(\mathbf{R}^n)$.

First note that $g \in H^k(\mathbf{R}^n)$ if and only if $\widehat{g} \cdot (1+|\xi|^2)^{\frac{k}{2}} \in L^2(\mathbf{R}^n)$. Note also that the Fourier transform of $(1-\Delta)$ is $1+4\pi^2|\xi|^2$, so the Fourier transform of $(1-\Delta)^3$ is $\left(1+4\pi^2|\xi|^2\right)^3$. Then, $\left(1+4\pi^2|\xi|^2\right)^3 \widehat{u} = \widehat{f}$, so

$$\widehat{u} = \frac{\widehat{f}}{(1+4\pi^2|\xi|^2)^3} \cdot \left(\frac{1+|\xi|^2}{1+|\xi|^2}\right)^3.$$

Since $f \in H^s(\mathbf{R}^n)$, $\widehat{f} \cdot (1+|\xi|^2)^{\frac{s}{2}} \in L^2(\mathbf{R}^n)$. Furthermore, $\frac{1+|\xi|^2}{(1+4\pi^2|\xi|^2)^3}$ is smooth and bounded, so $\left(\frac{1+|\xi|^2}{1+4\pi^2|\xi|^2}\right)^3$ is. Then, observe as

$$\begin{aligned}
\widehat{u} \cdot \left(1+|\xi|^2\right)^{\frac{s+6}{2}} &= \frac{\widehat{f}}{(1+4\pi^2|\xi|^2)^3} \cdot \left(\frac{1+|\xi|^2}{1+|\xi|^2}\right)^3 \cdot \left(1+|\xi|^2\right)^{\frac{s+6}{2}} \\
&= \left(\frac{1+|\xi|^2}{1+4\pi^2|\xi|^2}\right)^3 \cdot \widehat{f} \cdot \frac{(1+|\xi|^2)^{\frac{s}{2}+3}}{(1+|\xi|^2)^3} \\
&= \left(\frac{1+|\xi|^2}{1+4\pi^2|\xi|^2}\right)^3 \cdot \widehat{f} \cdot (1+|\xi|^2)^{\frac{s}{2}}.
\end{aligned}$$

Therefore, $\widehat{u} \cdot (1+|\xi|^2)^{\frac{s+6}{2}} \in L^2(\mathbf{R}^n)$, and thus, $u \in H^{s+6}(\mathbf{R}^n)$, as desired.

10. (4/9)

Let $a \in \mathbf{R}$. Compute the Fourier transform of $e^{-a\pi x^2}$ where $x \in \mathbf{R}$.

We compute $\widehat{f}(\xi) = \int_{\mathbf{R}} e^{-2\pi i x \cdot \xi} f(x) dx$. If $f(x) = e^{-a\pi x^2}$, then $f'(x) = -2a\pi x e^{-a\pi x^2} = -2a\pi x f(x)$.

Thus, take the Fourier transform of both sides:

$$\begin{aligned}
\widehat{(f')}(\xi) &= \widehat{(-2a\pi\xi f)}(\xi) \\
2\pi i\xi \widehat{f}(\xi) &= -2a\pi \cdot \frac{i}{2\pi}\left(\widehat{f}(\xi)\right)' \\
2\pi i\xi \widehat{f}(\xi) &= -ai\left(\widehat{f}(\xi)\right)' \\
\frac{-2\pi}{a}\xi \widehat{f}(\xi) &= \left(\widehat{f}(\xi)\right)'.
\end{aligned}$$

Thus, $\widehat{f}(\xi) = C\exp\left(\frac{-2\pi}{a}\cdot\frac{\xi^2}{2}\right)$, and $C = \widehat{f}(0) = \int_{\mathbf{R}} f(x)dx = \int_{\mathbf{R}} e^{-a\pi x^2}dx = \frac{1}{\sqrt{a}}$. Therefore,

$$\widehat{f}(\xi) = \frac{1}{\sqrt{a}}e^{\frac{-\pi\xi^2}{a}}.$$

2.2 Aug 2014

I. Laplace Equation

1. **2.1.1**
2. **2.1.2**
3. (6/9)

> Let $\Omega \subseteq \mathbf{R}^n$ be a bounded smooth domain. Let $\lambda \in \mathbf{R}$ and $u \in C^2\left(\overline{\Omega}\right)$ such that $\Delta u = \lambda u$ on Ω and $u|_{\partial\Omega} = 0$. Prove that either $u = 0$ on Ω or $\lambda < 0$.

By Green's Theorem,

$$\int_\Omega \nabla u \cdot \nabla u dx = -\int_\Omega u\Delta u dx + \int_{\partial\Omega} u \cdot \frac{du}{d\nu}d\sigma = -\int_\Omega u\cdot\lambda u dx + \int_{\partial\Omega} u\cdot\frac{du}{d\nu}d\sigma.$$

Since $u \equiv 0$ on $\partial\Omega$, $\int_\Omega \nabla u\cdot\nabla u dx = -\int_\Omega \lambda u^2 dx$. Thus, $\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right)dx = 0$. If $\lambda < 0$ there is nothing to show, so assume $\lambda \geq 0$ and consider the following cases.

If $\lambda > 0$, then $\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right)dx = 0$ for all x forces $|\nabla u|^2 + \lambda u^2 = 0$. Everything is nonnegative, forcing $u \equiv 0$.

If $\lambda = 0$, then $\Delta u = 0$, so u is harmonic. By the maximum and minimum principles, u attains its max and min on $\partial\Omega$, so $\max u = \min u = 0$, so $u \equiv 0$.

Therefore, either $\lambda < 0$, or $u \equiv 0$.

II. Heat Equations

4. **2.1.5**
5. (4/9)

> Suppose that c is a constant and u is a smooth solution of
>
> $$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbf{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}, \end{cases}$$
>
> where $f \in C_C\left(\mathbf{R}^n \times [0,\infty)\right)$ and $g \in C_C\left(\mathbf{R}^n\right)$. Show that there exists a constant $C > 0$ so that
>
> $$|u(x,t)| < Ce^{-ct}.$$

We use integration factors. If $u_t - \Delta u + cu = f$, then $u_t e^{ct} - \Delta u e^{ct} + cue^{ct} = fe^{ct}$. Notice that since $(ue^{ct})_t = u_t e^{ct} + uce^{ct}$ and since $\Delta u e^{ct} = \Delta(ue^{ct})$ as $\Delta = \Delta_x$, we get $(ue^{ct})_t - \Delta(ue^{ct}) = e^{ct} f$.

We have thus shown that ue^{ct} must solve the non-homogeneous heat equation, and therefore it is well-known that

$$ue^{ct} = \int_0^t \int_{\mathbf{R}^n} \Phi(x-y, t-s) \cdot e^{ct} f(y,s) dyds + \int_{\mathbf{R}^n} \Phi(x-y,t) \cdot g(y) dy,$$

where

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \cdot e^{\frac{-|x|^2}{4t}}$$

is the fundamental solution of the heat equation. (Note that *what* Φ is does not matter here, only that $\displaystyle\int_{\mathbf{R}^n} \Phi(x,t)dx = 1$.)

Now, since f and g are compactly supported, we know that $|e^{ct} \cdot f| \leq M$ and $|g| \leq N$ for some $M, N > 0$. By the triangle inequality and since $\Phi \geq 0$,

$$\begin{aligned}
\left|ue^{ct}\right| &\leq \int_0^t \int_{\mathbf{R}^n} |\Phi(x-y,t-s)| \cdot \left|e^{ct} \cdot f\right| dyds + \int_{\mathbf{R}^n} |\Phi(x-y,t)| \cdot |g| dy \\
&\leq M \int_0^t \int_{\mathbf{R}^n} \Phi(x-y,t-s) dyds + N \int_{\mathbf{R}^n} \Phi(x-y,t) dy \\
&= M \int_0^t ds + N \\
&= Mt + N.
\end{aligned}$$

Thus, $|u| \leq (Mt+N)e^{-ct}$. For $t > 0$, $(Mt+N)e^{-ct}$ is bounded. Thus it reaches a maximum at some $t_0 > 0$. Let $C = Mt_0 + N$; then $|u(x,t)| \leq Ce^{-ct}$, as desired.

III. Wave Equations

6. (5/9)

> Let u solve the initial value problem for the wave equation in one dimension:
>
> $$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbf{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbf{R} \times \{t = 0\}. \end{cases}$$
>
> Suppose g and h have compact support. The kinetic energy is $k(t) = \dfrac{1}{2} \displaystyle\int_{-\infty}^{\infty} {u_t}^2(x,t)dx$ and the potential energy is $p(t) = \dfrac{1}{2} \displaystyle\int_{-\infty}^{\infty} {u_x}^2(x,t)dx$. Prove $k(t) = p(t)$ for all large enough times t.

7. **2.1.7**

IV. Miscellaneous

8. **2.1.8**

9. **2.1.9**

10. (5/9)

> Let $\varphi(x) = \frac{1}{2}e^{-|x|}$ on $\mathbf{R}$. Use the Fourier transform to derive the solution $u = f * \varphi$ of the differential equation $u - u'' = f$.

Note that

$$\begin{aligned}
u(x,t) &= \int_{\mathbf{R}} e^{2\pi i x\cdot\xi}\widehat{u}(\xi,t)d\xi, \text{ so}\\
(1-\Delta)u &= \int_{\mathbf{R}} e^{2\pi i x\cdot\xi}(1+4\pi^2|\xi|^2)\widehat{u}d\xi.\\
\text{Also, } f &= \int_{\mathbf{R}} e^{2\pi i x\cdot\xi}\widehat{f}d\xi.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= (1-\Delta)u - f\\
0 &= \int_{\mathbf{R}} e^{2\pi i x\cdot\xi}\left((1+4\pi^2|\xi|^2)\widehat{u} - \widehat{f}\right)d\xi.
\end{aligned}$$

This forces

$$\begin{aligned}
0 &= (1+4\pi^2|\xi|^2)\widehat{u} - \widehat{f}\\
\widehat{u} &= \widehat{f}\cdot(1+4\pi^2|\xi|^2)^{-1}\\
u &= f * \left((1+4\pi^2|\xi|^2)^{-1}\right)^{\vee}.
\end{aligned}$$

From the other direction,

$$\begin{aligned}
\widehat{\varphi} &= \int_{\mathbf{R}} e^{-2\pi i x\cdot\xi}\varphi(x)dx\\
&\int_{\mathbf{R}} e^{-2\pi i x\cdot\xi}\frac{1}{2}e^{-|x|}dx\\
&= \frac{1}{2}\int_{-\infty}^{0} e^{x(1-2\pi i\xi)}dx + \frac{1}{2}\int_{0}^{\infty} e^{-x(1+2\pi i\xi)}dx\\
&= \frac{1}{2}\left.\frac{e^{x(1-2\pi i\xi)}}{1-2\pi i\xi}\right|_{-\infty}^{0} + \frac{1}{2}\left.\frac{e^{-x(1+2\pi i\xi)}}{1+2\pi i\xi}\right|_{0}^{\infty}\\
&= \frac{1}{2}\left(\frac{1}{1-2\pi i\xi} - 0 - 0 + \frac{1}{1+2\pi i\xi}\right)\\
&= \frac{1}{2}\left(\frac{1+2\pi i\xi+1-2\pi i\xi}{(1-2\pi i\xi)(1+2\pi i\xi)}\right)\\
&= \frac{1}{1+4\pi^2|\xi|^2}.
\end{aligned}$$

So $\varphi = \left((1+4\pi^2|\xi|^2)^{-1}\right)^{\vee}$, and therefore $u = f*\varphi$, as desired.

2.3 Jan 2015

I. Laplace Equation

1. **2.2.3**
2. **2.1.2**
3. **2.1.3**

II. Heat Equations

4. **2.2.5**
5. **2.1.5**

III. Wave Equations

6. (5/9)

> Suppose that u is a C^2 solution of the wave equation in $\mathbf{R}^n \times \mathbf{R}$. Show that if $u(\cdot, t_0)$ and $u_t(\cdot, t_0)$ have compact support in $\mathbf{R}^n$ for some t_0, then $u(\cdot, t)$ has compact support in $\mathbf{R}^n$ for all $t \geq 0$.

Pick $x_1 \in \mathbf{R}^n$, and let $R > 0$. Pick $t_0 > 0$ and $x_0 \in \mathbf{R}^n \setminus \overline{B(x_1, R+t_0)}$. Notice that $\overline{B(x_1, R)} \cap \overline{B(x_0, t_0)} = \emptyset$, so $\overline{B(x_0, t_0)} \subseteq \mathbf{R}^n \setminus \overline{B(x_1, R)}$. Let $C(x_0, t_0) = \{(x, t) \mid 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$. Now, $u \equiv 0$ in $C(x_0, t_0)$, so $u(x, t_0) = 0$ for all $x \in \mathbf{R}^n \setminus \overline{B(x_1, R+t_0)}$, and u has compact support at t_0 in $\mathbf{R}^n$. But t_0 was arbitrary, so $u(\cdot, t)$ has compact support for all $t \geq 0$.

7. (3/9)

> Derive a d'Alembert-like formula for u if $a > 0$ and u solves the equation
>
> $$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & \text{in } \mathbf{R} \times (0, \infty) \\ u(x, 0) = g(x) & \text{on } \mathbf{R} \times \{t = 0\} \\ u_t(x, 0) = h(x) & \text{on } \mathbf{R} \times \{t = 0\} \end{cases}$$
>
> where g is continuous and h is continuously differentiable.

As with d'Alembert, assume $u(x, t) = F(x + at) + G(x - at)$. Then $u(x, 0) = F(x) + G(x) = g(x)$, and $u_t(x, 0) = aF_t(x) - aG_t(x) = h(x)$, so $aF(x) - aG(x) = \displaystyle\int_{-\infty}^{x} h(y)dy$. Thus, we have the system

$$\begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \dfrac{1}{a} \displaystyle\int_{-\infty}^{x} h(y)dy. \end{cases}$$

Solving for F and G,

$$\begin{aligned} 2F(x) &= g(x) + \frac{1}{a} \int_{-\infty}^{x} h(y)dy \\ F(x) &= \frac{1}{2} g(x) + \frac{1}{2a} \int_{-\infty}^{x} h(y)dy; \\ 2G(x) &= g(x) - \frac{1}{a} \int_{-\infty}^{x} h(y)dy \\ G(x) &= \frac{1}{2} g(x) - \frac{1}{2a} \int_{-\infty}^{x} h(y)dy. \end{aligned}$$

Thus,

$$\begin{aligned} u(x, t) &= F(x + at) + G(x - at) \\ &= \frac{1}{2} g(x + at) + \frac{1}{2a} \int_{-\infty}^{x+at} h(y)dy + \frac{1}{2} g(x - at) - \frac{1}{2a} \int_{-\infty}^{x-at} h(y)dy \\ &= \frac{1}{2} \Big(g(x + at) + g(x - at) \Big) + \frac{1}{2a} \int_{x-at}^{x+at} h(y)dy. \end{aligned}$$

IV. Miscellaneous

8. (5/9)

The problem has two parts.

(a) Solve the partial differential equation

$$-yu_x + xu_y - u_z = u, \text{ for } (x, y) \in \mathbf{R}^2 \text{ and } z > 0$$
$$u(x, y, 0) = x - y, \text{ for } (x, y) \in \mathbf{R}^2.$$

(b) Prove that your solution solves the PDE.

(a) Let $x' = -y$, $y' = x$, $z' = -1$, and $f'(t) = f(t)$. Then we have to solve

$$X' = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X.$$

This is a well known ODE with solution $x = c_1 \cos(-t) + c_2 \sin(-t)$ and $y = c_2 \cos(-t) - c_1 \sin(-t)$. And clearly, $z = -t + c_3$ and $f(t) = c_4 e^t$.
When $t = 0$, $z = 0$, so $c_3 = 0$ and thus $t = -z$. So $x = c_1 \cos(z) + c_2 \sin(z)$ and $y = c_2 \cos(z) - c_1 \sin(z)$. We now solve for c_1 and c_2. See that

$$c_1 = \frac{x - c_2 \sin(z)}{\cos(z)},$$

so

$$\begin{aligned} y &= c_2 \cos(z) - \frac{x - c_2 \sin(z)}{\cos(z)} \cdot \sin(z) \\ y \cos(z) &= c_2 \cos^2(z) - x \sin(z) + c_2 \sin^2(z) \\ y \cos(z) + x \sin(z) &= c_2. \end{aligned}$$

Thus

$$\begin{aligned} c_1 &= \frac{x - (y \cos(z) + x \sin(z)) \sin(z)}{\cos(z)} \\ &= \frac{x - y \cos(z) \sin(z) - x \sin^2(z)}{\cos(z)} \\ &= \frac{x - y \cos(z) \sin(z) - x(1 - \cos^2(z))}{\cos(z)} \\ &= \frac{-y \cos(z) \sin(z) + x \cos^2(z)}{\cos(z)} \\ &= x \cos(z) - y \sin(z). \end{aligned}$$

Now, $c_4 = c_1 - c_2 = x \cos(z) - y \sin(z) - y \cos(z) - x \sin(z)$. Therefore,

$$u(x, y, z) = \Big(x \cos(z) - y \sin(z) - y \cos(z) - x \sin(z)\Big) e^{-z}.$$

(b) First, see that $u(x,y,0) = x - y$, as desired. Next, see that

$$\begin{aligned}
u_x &= \Big(\cos(z) - \sin(z)\Big)e^{-z}, \\
u_y &= \Big(-\sin(z) - \cos(z)\Big)e^{-z}, \text{ and} \\
u_z &= \Big(-x\sin(z) - y\cos(z) + y\sin(z) - x\cos(z)\Big)e^{-z} \\
&\quad - \Big(x\cos(z) - y\sin(z) - y\cos(z) - x\sin(z)\Big)e^{-z} \\
&= \Big(2y\sin(z) - 2x\cos(z)\Big)e^{-z},
\end{aligned}$$

so

$$-yu_x + xu_y - u_z = \Big(-y\cos(z) - y\sin(z) - x\sin(z) + x\cos(z)\Big)e^{-z} = u,$$

as desired.

9. **2.1.9**
10. **2.1.10**

2.4 Aug 2015

I. Laplace Equation

1. **2.2.3**
2. (4/9)

> Let (u_n) be a sequence of harmonic functions on an open set Ω, and suppose that $u_1(x) \le u_2(x) \le u_3(x) \le \cdots$ for all $x \in \Omega$. Suppose that
>
> $$\lim_{n\to\infty} u_n(x_0)$$
>
> converges for some fixed point $x_0 \in \Omega$. Prove that u_n converges uniformly on compact subsets of Ω to a harmonic function u on Ω.

Let $k > j$, and let $f = u_k - u_j$. Then f is a difference of harmonic functions, hence harmonic, and moreover, since $k > j$, $f \ge 0$. Thus, on each connected open set $U \subset\subset \Omega$, by Harnack's inequality there exists a positive constant $C = C(U)$ such that $\sup_U f \le C \inf_U f$. Furthermore, since $\lim_{n\to\infty} u_n(x_0)$ converges, for sufficiently large k and j, $|u_k(x_0) - u_j(x_0)| = f(x_0) < \frac{\varepsilon}{C}$. Thus,

$$0 \le \sup_U f \le C \inf_U f \le Cf(x_0) < \varepsilon.$$

Hence, $f \to 0$ uniformly on compact subsets U of Ω, so u_n is uniformly Cauchy, hence converges uniformly to a limit function u on Ω. Furthermore, by a corollary to the converse to the mean value property, since (u_n) is a sequence of harmonic functions on Ω that converges uniformly on compact sets of Ω to a limit u, u is harmonic on Ω.

3. **2.1.3**

II. Heat Equations

4. **2.2.5**

5. (1/9)

Consider the function $u(x,t) = xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}$. Show that

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbf{R} \text{ and } t > 0 \\ \lim\limits_{t\to 0^+} u(x,t) = 0, & x \in \mathbf{R} \end{cases}$$

Why is this surprising?

A straightforward, if tedious, computation:

$$\begin{aligned}
u_t - \Delta u &= \frac{\partial}{\partial t}\left[xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\right] - \frac{\partial^2}{\partial x^2}\left[xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\right] \\
&= x\frac{-3}{2}t^{\frac{-5}{2}}e^{\frac{-x^2}{4t}} + xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{x^2}{4t^2} - \frac{\partial}{\partial x}\left[t^{\frac{-3}{2}}e^{\frac{-x^2}{4t}} + xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{-2x}{4t}\right] \\
&= x\frac{-3}{2}t^{\frac{-5}{2}}e^{\frac{-x^2}{4t}} + xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{x^2}{4t^2} - \left(t^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{-2x}{4t} + t^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{-2x}{4t}\right. \\
&\quad \left. + xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{-2x}{4t}\frac{-2x}{4t} + xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}}\frac{-2}{4t}\right) \\
&= \frac{-3x}{2t^{\frac{5}{2}}e^{\frac{x^2}{4t}}} + \frac{x^3}{4t^{\frac{7}{2}}e^{\frac{x^2}{4t}}} + \frac{x}{2t^{\frac{5}{2}}e^{\frac{x^2}{4t}}} + \frac{x}{2t^{\frac{5}{2}}e^{\frac{x^2}{4t}}} - \frac{x^3}{4t^{\frac{7}{2}}e^{\frac{x^2}{4t}}} + \frac{x}{2t^{\frac{5}{2}}e^{\frac{x^2}{4t}}} \\
&= 0.
\end{aligned}$$

And by iterations of l'Hôpital,

$$\lim_{t\to 0^+} xt^{\frac{-3}{2}}e^{\frac{-x^2}{4t}} = \lim_{t\to 0^+}\frac{xt^{\frac{-3}{2}}}{e^{\frac{x^2}{4t}}} = \lim_{t\to 0^+}\frac{x\frac{-3}{2}t^{\frac{-5}{2}}}{e^{\frac{x^2}{4t}}\frac{-x^2}{4t^2}} = \lim_{t\to 0^+}\frac{6t^{\frac{-1}{2}}}{xe^{\frac{x^2}{4t}}} = \lim_{t\to 0^+}\frac{6\frac{-1}{2}t^{\frac{-3}{2}}}{xe^{\frac{x^2}{4t}}\frac{-x^2}{4t^2}} = \lim_{t\to 0^+}\frac{12t^{\frac{1}{2}}}{x^3e^{\frac{x^2}{4t}}} = 0.$$

This is surprising because u is not unique; the zero function also satisfies the differential equation.

III. Wave Equations

6. **2.3.6**

7. (2/9)

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded smooth domain. Let $\lambda \in \mathbf{R}$ and $g \in C^2\left(\overline{\Omega}\right)$ such that $\Delta g = \lambda g$ and $g|_{\partial\Omega} = 0$. Find a solution $u(x,t)$ to the wave equation $u_{tt} - \Delta u = 0$ on $\overline{\Omega} \times [0,\infty)$ satisfying $u(x,0) = g(x)$, $u_t(x,0) = 0$, and $u|_{\partial\Omega} = 0$ for all $t \geq 0$.

Assume that $u(x,t) = h(x)f(t)$; that is, that u is separable. Then $u_{tt} = h(x)f''(t)$ and $\Delta u = \Delta h(x)f(t)$. Furthermore, if $u_t(x,0) = 0$, then $f'(0) = 0$. To build our solution, first let $h(x) = g(x)$. Then $u_{tt} - \Delta u = 0$, so $g(x)f''(t) - \Delta g(x)f(t) = 0$, so $g(x)f''(t) = \Delta g(x)f(t)$, so $\frac{f''(t)}{f(t)} = \frac{\Delta g(x)}{g(x)} = \frac{\lambda g(x)}{g(x)} = \lambda$, and therefore $f(t) = c_1e^{\sqrt{\lambda}t} + c_2e^{-\sqrt{\lambda}t}$. We may conveniently rewrite this as

$$\begin{aligned}
f(t) &= c_1e^{i\sqrt{-\lambda}t} + c_2e^{-i\sqrt{-\lambda}t} \\
&= c_1\left(\cos(\sqrt{-\lambda}t) + i\sin(\sqrt{-\lambda}t)\right) + c_2\left(\cos(-\sqrt{-\lambda}t) + i\sin(-\sqrt{-\lambda}t)\right) \\
&= c_1\cos(\sqrt{-\lambda}t) + c_1i\sin(\sqrt{-\lambda}t) + c_2\cos(-\sqrt{-\lambda}t) + c_2i\sin(-\sqrt{-\lambda}t) \\
&= c_1\cos(\sqrt{-\lambda}t) + c_1i\sin(\sqrt{-\lambda}t) + c_2\cos(\sqrt{-\lambda}t) - c_2i\sin(\sqrt{-\lambda}t) \\
&= (c_1+c_2)\cos(\sqrt{-\lambda}t) + (c_1-c_2)i\sin(\sqrt{-\lambda}t).
\end{aligned}$$

Now we use the fact that $f'(0) = 0$ to solve for c_1 and c_2. See that

$$f'(t) = -(c_1 + c_2)\sin(\sqrt{-\lambda}t)\sqrt{-\lambda} + (c_1 - c_2)i\cos(\sqrt{-\lambda}t)\sqrt{-\lambda}, \text{ so}$$
$$0 = -(c_1 + c_2)\cdot 0 \cdot \sqrt{-\lambda} + (c_1 - c_2)i \cdot 1 \cdot \sqrt{-\lambda},$$

so $c_1 - c_2 = 0$. Therefore $f(t) = (c_1 + c_2)\cos(\sqrt{-\lambda}t)$. And since $u(x, 0)$ must be $g(x)$, we have that $u(x, 0) = g(x)f(0) = g(x)(c_1 + c_2)\cos(0) = g(x)(c_1 + c_2)$, and therefore $c_1 + c_2 = 1$. This means that $u(x, t) = g(x)\cos(\sqrt{-\lambda}t)$.

IV. Miscellaneous

8. (Warning: **2.3.8**, with a +!)

The problem has two parts.

(a) Solve the partial differential equation

$$-yu_x + xu_y - u_z = u, \text{ for } (x, y) \in \mathbf{R}^2 \text{ and } z > 0$$
$$u(x, y, 0) = x + y, \text{ for } (x, y) \in \mathbf{R}^2.$$

(b) Prove that your solution solves the PDE.

(a) Let $x' = -y$, $y' = x$, $z' = -1$, and $f'(t) = f(t)$. Then we have to solve

$$X' = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X.$$

This is a well known ODE with solution $x = c_1\cos(-t) + c_2\sin(-t)$ and $y = c_2\cos(-t) - c_1\sin(-t)$. And clearly, $z = -t + c_3$ and $f(t) = c_4e^t$.

When $t = 0$, $z = 0$, so $c_3 = 0$ and thus $t = -z$. So $x = c_1\cos(z) + c_2\sin(z)$ and $y = c_2\cos(z) - c_1\sin(z)$. We now solve for c_1 and c_2. See that

$$c_1 = \frac{x - c_2\sin(z)}{\cos(z)},$$

so

$$\begin{aligned} y &= c_2\cos(z) - \frac{x - c_2\sin(z)}{\cos(z)} \cdot \sin(z) \\ y\cos(z) &= c_2\cos^2(z) - x\sin(z) + c_2\sin^2(z) \\ y\cos(z) + x\sin(z) &= c_2. \end{aligned}$$

Thus

$$\begin{aligned} c_1 &= \frac{x - (y\cos(z) + x\sin(z))\sin(z)}{\cos(z)} \\ &= \frac{x - y\cos(z)\sin(z) - x\sin^2(z)}{\cos(z)} \\ &= \frac{x - y\cos(z)\sin(z) - x(1 - \cos^2(z))}{\cos(z)} \\ &= \frac{-y\cos(z)\sin(z) + x\cos^2(z)}{\cos(z)} \\ &= x\cos(z) - y\sin(z). \end{aligned}$$

Now, $c_4 = c_1 + c_2 = x\cos(z) - y\sin(z) + y\cos(z) + x\sin(z)$. Therefore,

$$u(x, y, z) = \Big(x\cos(z) - y\sin(z) + y\cos(z) + x\sin(z)\Big)e^{-z}.$$

(b) First, see that $u(x,y,0) = x + y$, as desired. Next, see that

$$\begin{aligned}
u_x &= \Big(\cos(z) + \sin(z)\Big)e^{-z}, \\
u_y &= \Big(-\sin(z) + \cos(z)\Big)e^{-z}, \text{ and} \\
u_z &= \Big(-x\sin(z) - y\cos(z) - y\sin(z) + x\cos(z)\Big)e^{-z} \\
&\quad - \Big(x\cos(z) - y\sin(z) + y\cos(z) + x\sin(z)\Big)e^{-z} \\
&= \Big(-2x\sin(z) - 2y\cos(z)\Big)e^{-z},
\end{aligned}$$

so

$$-yu_x + xu_y - u_z = \Big(y\cos(z) - y\sin(z) + x\sin(z) + x\cos(z)\Big)e^{-z} = u,$$

as desired.

9. **2.1.9**
10. **2.2.10**

2.5 Jan 2016

I. Laplace Equation

1. **2.2.3**
2. **2.4.2**
3. **2.1.1**

II. Heat Equations

4. **2.1.4**
5. **2.1.5**

III. Wave Equations

6. **2.2.6**
7. **2.4.7**

IV. Miscellaneous

8. **2.1.8**
9. **2.1.9**
10. **2.1.10**

2.6 Aug 2016

I. Laplace Equation

1. **2.1.1**
2. **2.1.2**
3. **2.1.3**

II. Heat Equations

4. (2/9)

This problem is in two parts:

(a) Derive a representation formula for the initial value problem:

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ u(x,0) = g(x) \in C_C(\mathbf{R}^n) & \text{in } \mathbf{R}^n \times \{t=0\}. \end{cases}$$

(b) Write down a representation formula for the PDE

$$\begin{cases} \partial_t u - \Delta u + cu = 0 & \text{in } \mathbf{R}^n \times (0,\infty) \\ u(x,0) = g(x) \in C_C^2(\mathbf{R}^n) & \text{in } \mathbf{R}^n \times \{t=0\}. \end{cases}$$

where $c \in \mathbf{R}$ is a fixed constant.

5. **2.1.5**

III. Wave Equations

6. **2.3.6**

7. **2.3.7**

IV. Miscellaneous

8. **2.4.8**

9. **2.1.9**

10. **2.2.10**

2.7 Jan 2017

I. Laplace Equation

1. **2.2.3**

2. **2.1.2**

3. (2/9)

Let $u \in C(\mathbf{R}^n)$ satisfy the mean value property

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u\, d\sigma$$

If $\varphi \in C_C^\infty(\mathbf{R}^n)$ is radial (so $\varphi(x) = \psi(|x|)$ for some smooth function ψ on $\mathbf{R}$) and satisfies $\int_{\mathbf{R}^n} \varphi = 1$ and $\varphi_\varepsilon(x) = \varepsilon^n \varphi(\varepsilon^{-1}x)$, show that if $\varepsilon > 0$, then

$$u(x) = \int_{\mathbf{R}^n} u(y)\varphi_\varepsilon(x-y)dy, \text{ for all } x \in \mathbf{R}^n.$$

II. Heat Equations

4. **2.1.4**
5. **2.6.4**

III. Wave Equations

6. **2.2.6**
7. **2.3.6**

IV. Miscellaneous

8. **2.3.8**
9. **2.1.9**
10. **2.2.10**

2.8 Aug 2017

I. Laplace Equation

1. **2.2.3**
2. **2.4.2**
3. **2.7.3**

II. Heat Equations

4. **2.1.4**
5. **2.1.5**

III. Wave Equations

6. **2.2.6**
7. **2.3.6**

IV. Miscellaneous

8. **2.1.8**
9. **2.1.9**
10. **2.1.10**

2.9 Jan 2018

I. Laplace Equation

1. **2.1.1**
2. **2.4.2**
3. **2.1.3**

II. Heat Equations

4. (1/9)

> Let $\Omega \subseteq \mathbf{R}^n$ be a bounded smooth domain. Let $\lambda \in \mathbf{R}$ and $g \in C^2\left(\overline{\Omega}\right)$ such that $\Delta g = \lambda g$ and $g|_{\partial\Omega} = 0$. Find a solution $u(x,t)$ to the heat equation $u_t - \Delta u = 0$ on $\overline{\Omega} \times [0,\infty)$ satisfying $u(x,0) = g(x)$ and $u|_{\partial\Omega} = 0$ for all $t \geq 0$.

Assume that $u(x,t)$ is separable, so $u(x,t) = h(x)f(t)$. Then $u_t(x,t) = h(x)f'(t)$ and $\Delta u(x,t) = \Delta h(x)f(t)$. Thus $0 = u_t - \Delta u = h(x)f'(t) - \Delta h(x)f(t)$, so $\Delta h(x)f(x) = h(x)f'(t)$, and $\frac{\Delta h(x)}{h(x)} = \frac{f'(t)}{f(t)}$. To build the solution, let $h(x) = g(x)$, and since $\Delta g(x) = \lambda g(x)$, $\frac{f'(t)}{f(t)} = \frac{\lambda g(x)}{g(x)} = \lambda$, so $f'(t) = \lambda f(t)$, and thus $f(t) = Ce^{\lambda t}$. Since $u(x,0) = g(x)$, $u(x,0) = g(x)Ce^0 = g(x)C$, so $C = 1$. Therefore, $u(x,t) = g(x)e^{\lambda t}$.

5. **2.2.5**

III. Wave Equations

6. **2.2.6**
7. **2.3.7**

IV. Miscellaneous

8. **2.4.8**
9. **2.1.9**
10. **2.2.10**